



Undergraduate Review

Volume 12

Article 18

2016

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Pasciuto, Nicholas (2016). The Mystery of the Non-Transitive Grime Dice. *Undergraduate Review*, 12, 107-115.
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The Mystery of the Non-Transitive Grime Dice

NICHOLAS PASCIUTO

In this paper we begin by studying a set of non-transitive dice, known as the Grime Dice, which function similar to the game of Rock-Paper-Scissors (RPS). Perhaps you remember this game as a way of deciding things at some point in time like who goes first or who gets the last cookie. Rock-Paper-Scissors is an example of a non-transitive game, which means that although Rock breaks Scissors and Scissors cuts Paper, unexpectedly Paper wraps Rock (some people say Paper “covers” Rock) creating a cycle of victory where no player has an advantage. The Grime dice function similar to this game with some differences. We observe various properties of these dice such as dual non-transitive chains and a counterintuitive, reversing property when moving from one die to two dice. We determine that this reversing property does not continue as the number of dice increases and provide some evidence that suggests an infinitely reversing characteristic may not be impossible. We examine other sets of non-transitive dice and constructed non-transitive sets of dice, including some new sets of our own creation, for all of the Platonic solids.

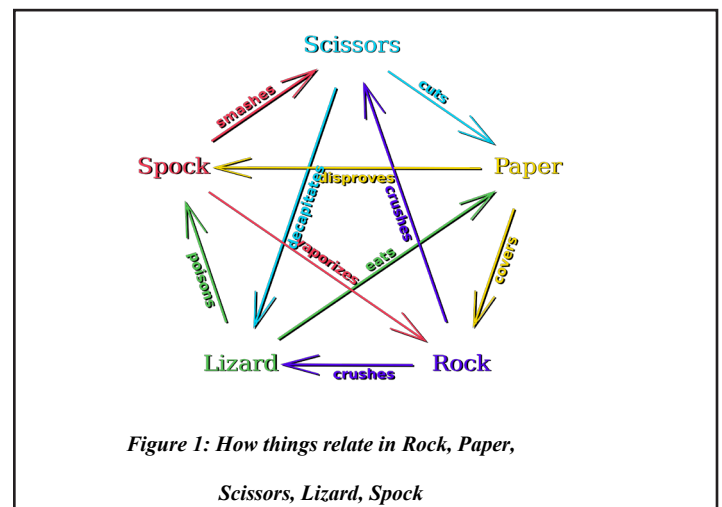
A relation on a set of objects is said to be transitive if, for every three objects, x, y, z , in the set, when x is related to y and y is related to z then x is related to z . For example, the relation, “greater than” (symbolized by $>$) is a transitive relation on the set of real numbers because for every three real numbers, a, b, c , if $a > b$ and $b > c$ then $a > c$. The term “transitive” means carries across. In the case of the relation “greater than” ($>$), the property of one object being greater than another carries across all the objects in the set from a to b to c , giving $a > c$.

A relation on a set of objects is non-transitive if, for every three objects, x, y, z , in the set, when x is related to y and y is related to z , then z is related to x . “Non-transitive” refers to the fact that the relation does not carry across from the first to the last object but in fact creates a cycle.

Both properties can be found in many real world situations. For example, a lynx preys on a fox, a fox preys on a squirrel, however, the lynx also preys on squirrels completing a transitive relation among these animals. A non-transitive chain occurring in nature is the California side-blotched male lizards whose different variations compete for female lizards with each male variation having an advantage against another in a non-transitive chain of three.

A familiar example of a non-transitive situation is the game of Rock, Paper, Scissors (RPS). In this game Rock beats Scissors, Scissors beat Paper, and Paper beats Rock. This non-transitive cycle is called a chain. Thus RPS contains the chain: Rock beats Scissors, Scissors beats Paper, and Paper beats Rock. Of course, each chain could start at any object because they are cycles.

Note the appearance of two distinct chains each of which includes all the objects in the set. The outside chain exists in the form of a circle while the inside chain exists in the form of a five pointed star. The outside chain: starting at Paper, which covers Rock, continues through Rock crushes Lizard, Lizard poisons Spock, Spock smashes Scissors, and completes the chain with Scissors cut Paper. The inside chain: starting with Scissors which decapitates Lizard,



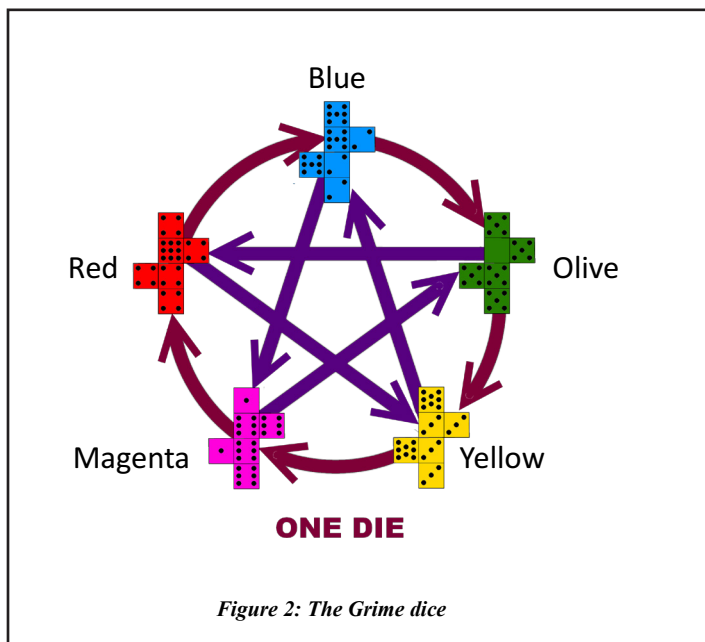
Lizard eats Paper, Paper disproves Spock, Spock vaporizes Rock, and concludes the chain with Rock crushes Scissors.

The game of RPS is played with two players throwing, at the same time, one of the hand signals for either Rock (a closed fist), Paper (an open, flat hand) or Scissors (two fingers separated in the shape of a pair of scissors). If two players, player A and player B, randomly choose which sign to throw then they will tie (throw the same sign) one third ($3/9$) of the time, player A will win one third of the time ($3/9$) and player B will win one third ($3/9$) of the time. Thus when there is a winner the probability of a player winning is 50%. When there are five objects (RPSLS) in the set, the two-player game is similar in terms of winning percentages because each object defeats exactly two others and loses to exactly two others. If two players, player A and player B, randomly chose which sign to throw then they will tie (throw the same sign) one fifth ($5/25$) of the time, player A will win two fifths of the time ($10/25$) and player B will win two fifths ($10/25$) of the time. If there are three players, although the probability of winning given the objects are chosen at random is still even, the game becomes more complex with more options and outcomes. In this version it is not only possible that all three players throw the same sign, but also there could be either transitive triples (Rock, Scissors, Lizard) or non-transitive chains (Spock, Scissors, Lizard). A non-transitive chain like Spock, Scissors, Lizard may be interpreted as a different kind of a tie, while a transitive triple chain like Rock, Scissors, Lizard would be a win for the player throwing Rock.

Dr. James Grime invented Grime dice, which are built on the traditional cubic dice. They are six sided (hexagonal or cubic) dice each of whose sides (or faces) contain any number of dots from 0 to 9. There are five dice colored Red, Blue, Olive, Yellow and Magenta. Figure 2 gives a representation of the dice. <http://singingbanana.com/dice/article.htm>

In Figure 2 each die is unfolded so that all six faces are visible in the plane. They are also arranged so that two distinct chains of who beats whom can be identified. The dice are cleverly designed so that the two non-transitive chains can be easily identified by the

names of the colors of the dice. The number of letters in each color organizes the first chain, which will be referred to as the outer chain throughout this paper. So this chain is Red (3 letters), Blue (4 letters), Olive (5 letters), Yellow (6 letters), and Magenta (7 letters). Note that Magenta wraps around to beat Red. The next chain, referred to as the inner chain throughout this paper, is organized alphabetically



by the first letter of each color. Thus, starting with the earliest letter (alphabetically) of a color of the dice, B, the chain goes Blue, Magenta, Olive, Red, Yellow, and note that Yellow wraps around to beat Blue. Grime dice are similar to the set Rock, Paper, Scissors, Lizard, Spock in that both contain five objects and both share two non-transitive chains. Arrows in Figure 2 indicate these chains.

Some differences between RPSLS and Grime dice are (a) how the game is played, and (b) most importantly, the fact that winning in Grime dice is probabilistic. (a) Grime dice require choosing and rolling dice instead of throwing hand gestures. Each one of the five dice has two different numbers on the six faces. For example, the Red die has five sides with four dots (a roll of 4) and one side with nine dots (a roll of 9). The game is traditionally played by having one person, the challenger, seek out a person to play, the opponent. Then

the opponent picks one die from the set of five colored dice. Once that is done the challenger picks from the remaining four dice. The notion behind this is that the challenger may be able to pick a die that appears to be more likely on average to beat the other die. Players then simultaneously roll the dice and compare the size of the rolls. The numbers of dots (the size of the roll) is used to decide which die wins that roll. The die with the higher number will win that roll. A game may consist of the best of three rolls or the best five rolls or the best of some number of rolls. The winner is the player who wins a majority of the rolls that are not ties. (b) In RPSLS, Rock always breaks Scissors. This never changes. However in Grime dice sometimes the Red die beats the Blue die and sometimes the Blue die beats the Red die. This probabilistic aspect creates some interesting

property holds in a chain) means that the winning order of that chain will reverse when doubles are used instead of single dice. In this case the outside chain has the reversing property, so the new order (when pairs of dice are rolled instead of single dice) becomes Magenta, Yellow, Olive, Blue, Red, and Red wraps around to beat Magenta. If the challenger now picks first and chooses Blue, the opponent unaware of the reversing property for doubles would pick the color die that he or she assumes would beat Blue, namely Red. The opponent would be correct if it weren't for the fact they were playing doubles and thus the opponent has picked the losing die (because with doubled dice, counter-intuitively, Blue beats Red more often).

In Figure 3 each row is labeled with a color, and the properties of the die with that color are listed in the other eight

Color	Average	Mode	1st #	Prob. 1st	Fraction	2nd #	Prob. 2nd	Fraction
Red	4.833	4	4	0.833	5/6	9	0.167	1/6
Blue	4.500	n/a	2	0.500	1/2	7	0.500	1/2
Olive	4.167	5	0	0.167	1/6	5	0.833	5/6
Yellow	4.667	3	3	0.667	2/3	8	0.333	1/3
Magenta	4.333	6	1	0.333	1/3	6	0.667	2/3

Figure 3: The Grime Dice Characteristics

features for the game.

In particular, these dice contain a unique property referred to as the order reversing property or reversing property. In a normal game the opponent picks one die out of the five and the challenger picks a different die (one of the remaining four; no duplicates are allowed). After several games, because the opponent in general loses more often, the opponent might become a little suspicious. Perhaps the opponent even figures out how the challenger keeps winning. Instead of not playing any more the opponent may try to turn the tables and have the challenger pick first. The challenger should agree and just to make it interesting suggest that they each choose two dice of the same color instead of just one. Amazingly enough, now the reversing property of the outside chain comes into play.

To say a chain has the reversing property (or, the reversing

columns. The column titled Average shows the average (mean value) roll for that die. For example, Olive had an average roll of 4.167, which was calculated by adding the results of each side (in this case 0, 5, 5, 5, 5, 5) and then dividing by the number of sides (6). The next column, Mode, tells us which result is most likely to occur. Since each of the numbers on the Blue die (2 and 7) is on three of the faces then there is no single mode because both two and seven occur the same number of times. The third column contains the smallest result on the die, which is followed in the next column by the probability of that number occurring and then by the probability represented as a fraction. The next three columns are the same for the other number on the die. This information is helpful because it allows us to evaluate who wins against whom.

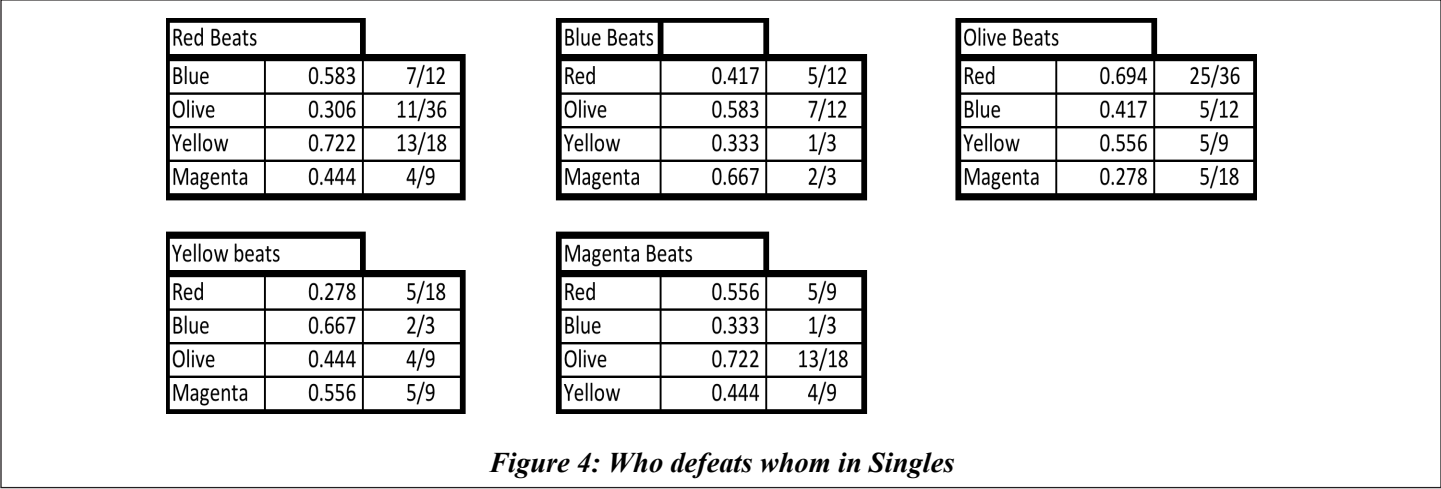
Figure 4 indicates the probability that a die would likely win

when faced against a different die.

There are five tables in this figure. Each table indicates how a particular die fares against each of the four other dice when rolled

based on the percentage from these tables.

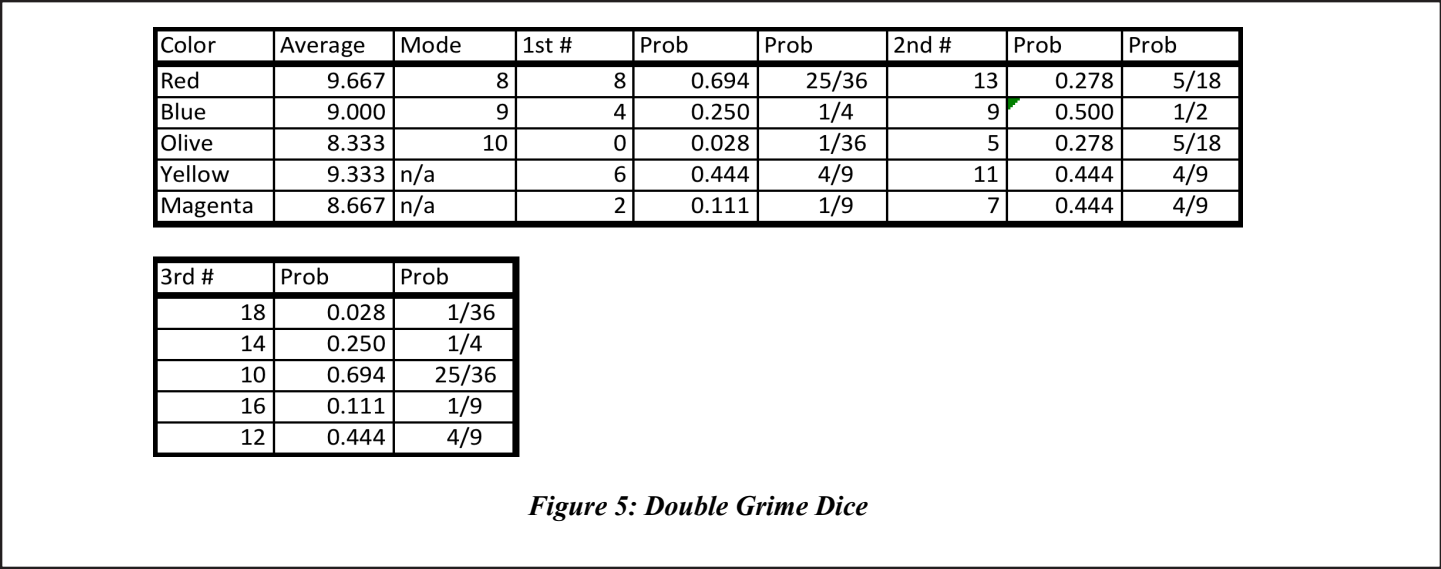
We now consider results from playing the game with pairs of dice of the same color.



once. The middle column of each table contains the probability in decimal form while the third column represents it as a fraction. Looking at the Red Table we see that Red beats both Blue and Yellow because the probability of rolling a higher number is greater than .50 (or 50%), but loses to both Magenta and Olive. Note that each die beats exactly two other dice and loses to exactly two other dice. Also observe that the highest winning percentage is 0.722 or (72.2%). These results are visually seen in Figure 2. The arrows are oriented

Figure 5: Double Grime Dice

Figure 5 reads the same as Figure 3. There is one specific change however, instead of only two numbers there are now three sums created by the sum of the numbers on the two dice. Using techniques of combinatorics we compute the number of ways of obtaining each of the sums, specifically using the formula for calculating the number of combinations of k things taken from a set



of n things, denoted by $\binom{n}{k}$ and read as “ n choose k ”. Next, probability trees were constructed to complete the probabilities for pairs of dice.

Figure 6 indicates the reversing property when playing with pairs of dice of the same color.

Figure 6: Who beats whom in Doubles

Figure 6 shows the results for a game with double dice.

and Magenta). Since there is still an outside chain, this indicates that the inside chain must break down. This was known about the dice; however, Grime argued that because Olive only loses to Red by a number (0.482 or 48.2%) which is very close to 50%, it can be considered as 50% and over a few rolls it will not significantly affect the outcome. So we may ignore its influence without great harm, assume there is an intact, inside chain, and in general the overall

Red		
Blue	0.410	59/144
Olive	0.518	423/817
Yellow	0.583	7/12
Magenta	0.691	56/81

Blue		
Red	0.590	85/144
Olive	0.410	59/144
Yellow	0.444	4/9
Magenta	0.556	5/9

Olive		
Red	0.482	231/479
Blue	0.590	85/144
Yellow	0.309	25/81
Magenta	0.389	7/18

Yellow		
Red	0.417	5/12
Blue	0.556	5/9
Olive	0.691	56/81
Magenta	0.407	11/27

Magenta		
Red	0.309	25/81
Blue	0.444	4/9
Olive	0.611	11/18
Yellow	0.593	16/27

Figure 6: Who beats whom in Doubles

Here, probability trees were constructed to complete the winning probabilities between pairs of dice. This is the same process albeit somewhat more complicated than what was used to determine the odds in Figure 4. Follow the original outside chain in reverse to see that the reversing property does indeed happen. However, drawing your attention to the Olive table we see a big problem. Olive no longer beats two other dice, but in fact loses to three (Red, Yellow,

probabilities will still work effectively for the challenger.

Figure 7 presents the results when three of each type of die are rolled. This figure is similar to Figures 3 and 5 with the difference that rolling three of each type of die gives four different possible sums.

The following Figure represents the likelihood of three dice all the same color winning against another different set of three dice

Color	Average	Mode	1st #	Prob	fraction	2nd #	Prob	fraction
Red	14.5	12	12	0.579	125/216	17	0.347	25/72
Blue	13.5	n/a	6	0.125	1/8	11	0.375	3/8
Olive	12.375	15	0	0.005	1/216	5	0.069	5/72
Yellow	14	14	9	0.296	8/27	14	0.444	4/9
Magenta	13	13	3	0.037	1/27	8	0.222	2/9

3rd #	Prob	fraction	4th #	Prob	fraction
22	0.069	5/72	27	0.005	1/216
16	0.375	3/8	21	0.125	1/8
10	0.347	25/72	15	0.579	125/216
19	0.222	2/9	24	0.037	1/27
13	0.444	4/9	18	0.296	8/27

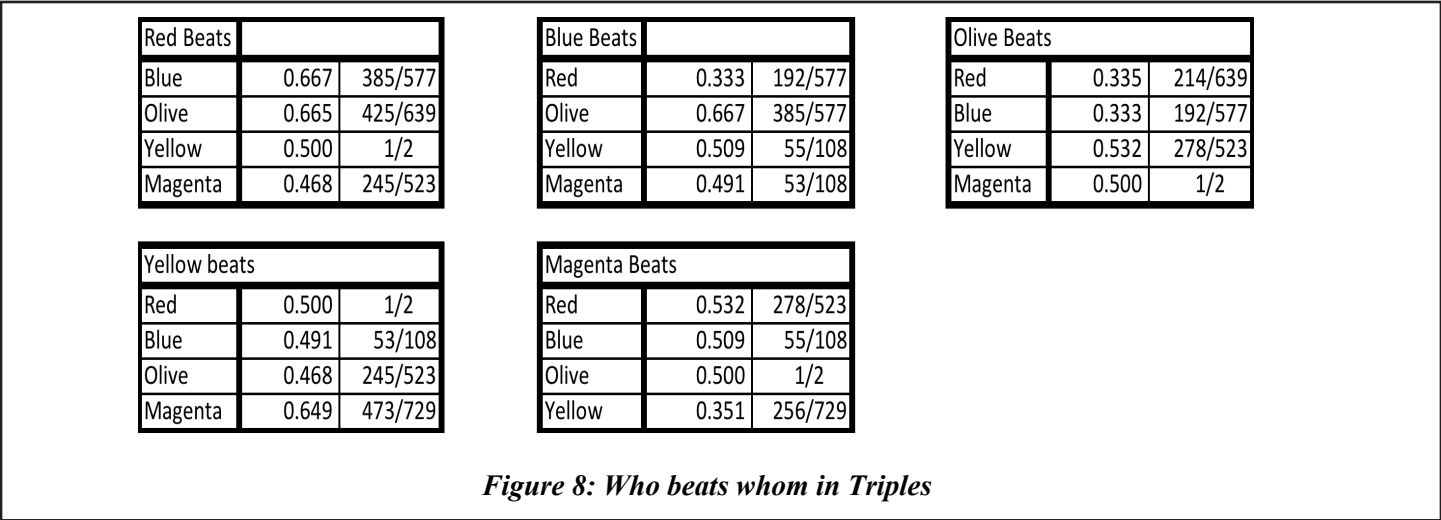
Figure 7: Triple Dice Results

all the same color.

Looking in Figure we can find that the original outside chain is reestablished. Now, for rolls of three dice, as in the game

and the dodecahedral dice. Finally we will conclude with sets of icosahedral dice.

Currently the only set of non-transitive tetrahedral dice



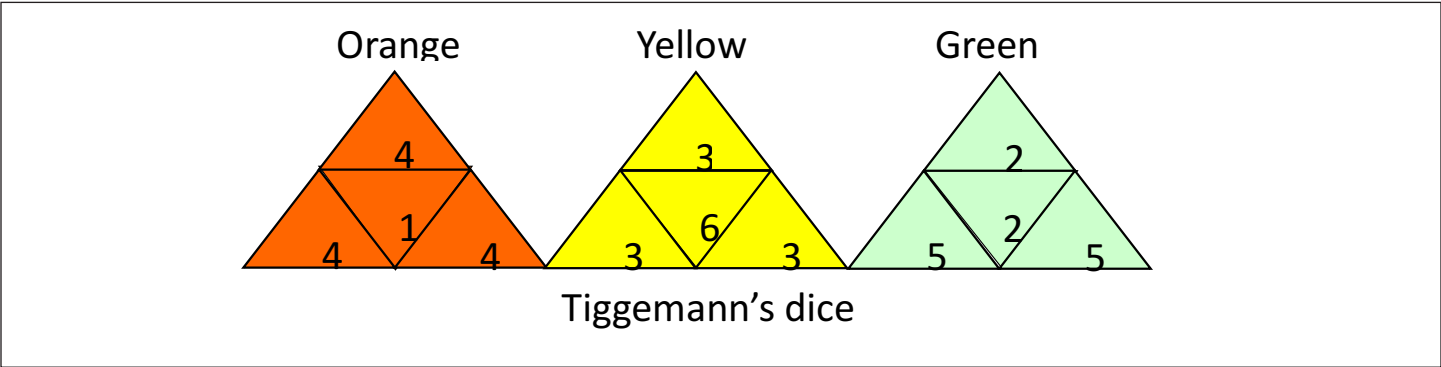
for where one die is rolled, we have again that Red beats Blue beats Olive beats Yellow beats Magenta beats Red. Thus the outside chain has actually reversed twice. However the inside chain is broken down even more than before.

In the rest of this paper we describe a Catalog of non-transitive dice for the five Platonic solids. Platonic solids are three-dimensional regular polygons, which means that every face of the solid is a regular polygon of the same size and shape. There is only one known set of non-transitive dice for the tetrahedral dice. After describing them we will move to discuss sets created for the hexahedral dice. Then explain which sets exist for octahedral dice

are ones designed by Daniel Tiggemann. He was curious to see if it was possible to create non-transitive dice for the tetrahedron. Here, Orange beats Yellow (9/16), Yellow beats Green (10/16) and Green beats Orange (10/16).

Tiggemann’s dice do in fact reverse. When doubled, Orange loses to Yellow (121/202), Yellow loses to Green (132/256) and Green loses to Orange (132/256).

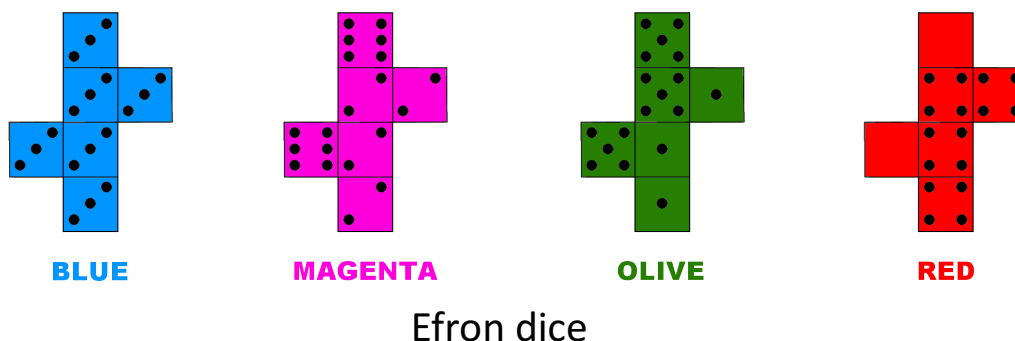
We used these dice to create sets for the octahedron (eight sides), the dodecahedron (twelve sides), and the icosahedron (twenty sides). It is possible to expand the tetrahedron to those solids because their number of sides are multiples of four. This means that since the



dice will retain the same probabilities, the dice will also have the same non-transitive cycles.

For the hexahedral dice there already exists the Grime dice, but there is also a set called the Efron dice. Brad Efron created the Efron dice, which uniquely have four objects in the set. Each die beats the next in the list by two thirds: Blue beats Magenta, Magenta beats Olive, Olive beats Red and Red beats Blue. This is different than all the other dice we have looked at because, in Efron dice, each die beats the other with the same probability. <http://singingbanana.com/dice/article.htm>

1,1,4,4,4,4,4,4 and 3,3,3,3,3,3,6,6 and 2,2,2,2,5,5,5,5. These have the same properties as the Tiggemann dice. The second set is a set of our own design. We began a search for a set of octahedral, non-transitive dice, which were not just an expansion of the tetrahedron. We worked on a set of three dice with three different numbers on each die and we continually modified the numbers until we obtained a non-transitive chain (Yellow->Orange->Green). Three numbers expanded the probabilities that we could use. We name this new set the Nichlman Dice by combining the first half of the author's first name with the last half of his mentor's last name. Oddly, the Yellow

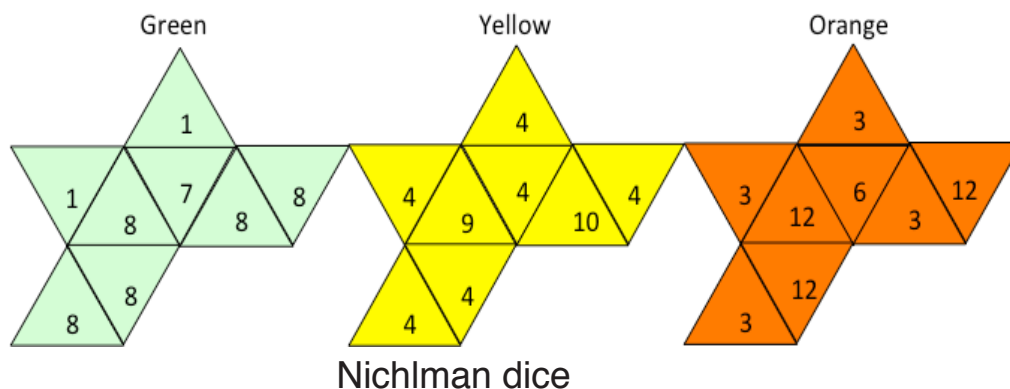


This set of dice was the first set of dice which we tested that did not reverse. They do not reverse because when doubled the Red die is too weak: it loses to both Blue and Olive. The Efron dice were used by Warren Buffet in a famous game with Bill Gates.

Currently there are two sets of octahedral dice. The first, as we previously stated, are the Tiggemann expanded to the octahedron;

die has the smallest average roll (5.25) yet it beats the die with the largest average roll, the Orange die (6.75).

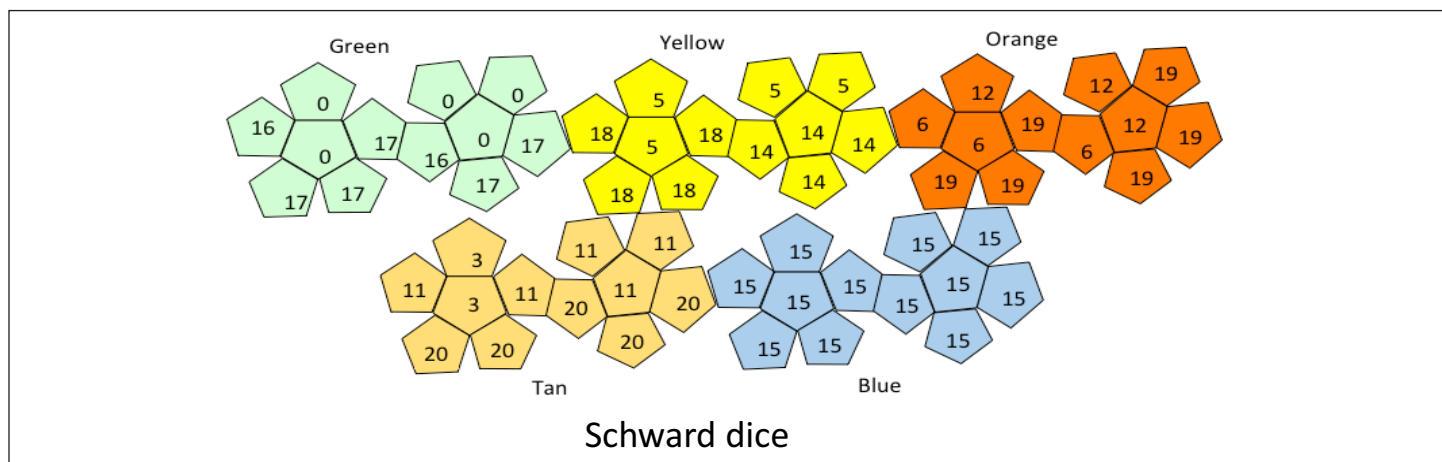
Although these dice form a non-transitive chain they do not in fact reverse because the Green die, which has the middle average (6.125), is too strong when doubled causing it to beat each of the other dice.



There are a total of three known sets for the dodecahedral dice. As previously stated we can create a set from expanding the Tiggemann dice. We created the second set of non-transitive dodecahedral dice and named them Schward by combining part of the author's last name and his mentor's first name. These dice only contain an outside chain and would need more work on the numbers and probabilities so that the dice will also have an inside chain.

expanding the Tiggemann dice. Another set of icosahedral dice is a set of dice that we created, which we named the Pascannell Dice. The name comes from the beginning part of the author's last name "Pas" combined with his girlfriend's last name "Scannell". These dice do not reverse.

Using the data collected from the Grime dice and trail and error we have been able to create new sets of non-transitive dice.

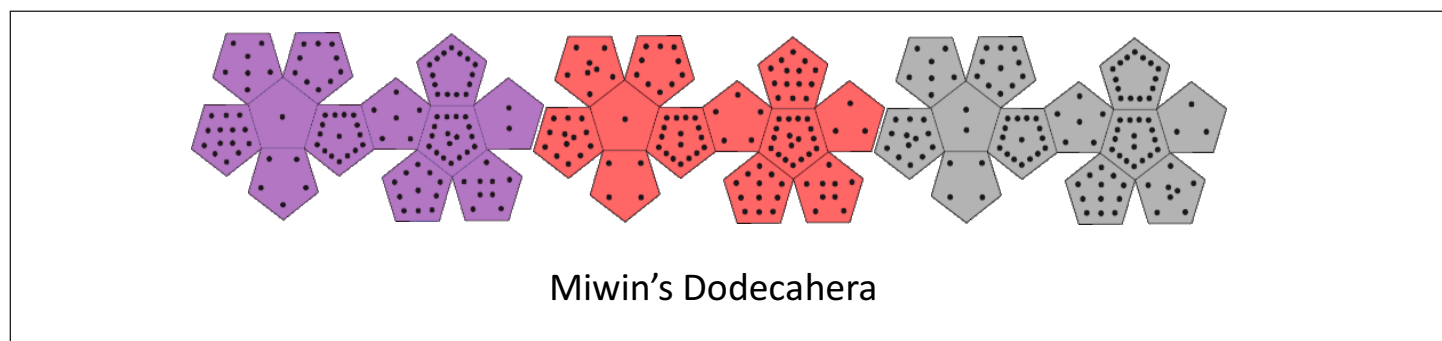


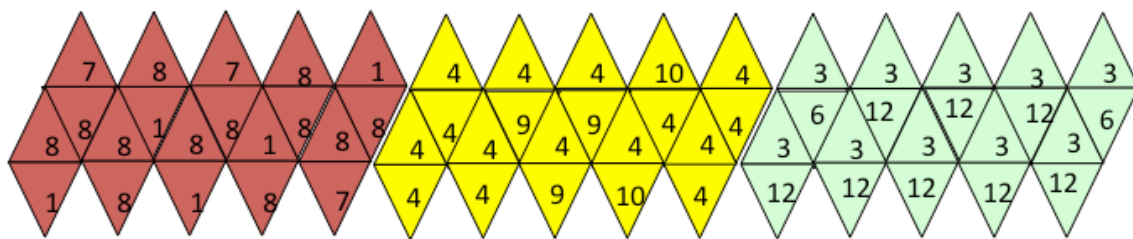
In addition to not having an inside chain the Schward dice do not reverse either.

Michael Winkleman invented a set of dice called Miwin's Dodecahedral dice, which also share a similar quality to the Efron dice. This property is that each die in the chain beats the next die with the same probability. https://en.wikipedia.org/wiki/Nontransitive_dice#Nontransitive_dodecahedra

The final platonic solid is the icosahedron. As previously stated we can create a set of non-transitive icosahedral dice by

This work shows that there are non-transitive dice not just for one Platonic solid but for all five of them. The study of non-transitive dice and patterns of non-transitive behavior has implications for many areas. A potentially important application of research of this type might be an investigation into whether non-transitive chains in nature or other fields might reverse as the number of animals or presence of materials increases beyond some point. For example, a food chain might reverse when the population's increase beyond some triggering point, or perhaps a subatomic process might invert





Pascannell dice

its actions when quantities increase beyond a critical measure. In any case, this counterintuitive reversibility in non-transitive situations will continue to fascinate people far into the future. The following questions are conjectures and areas for further research:

1. Do the measures of central tendency (mean, mode, median) give any indication of the behavior of the dice as the limit of the number of dice approaches infinity?
2. Compute an absolute average (Absolute average = $(\text{mean} + \text{mode} + \text{median})/3$) for each die and does the absolute average give any indication about what happens in general?
3. Is it possible to find a set of dice that reverses continually as the number of dice increases?
4. Is there a way to refine the definition of the reversing property so that we can make a general statement about non-transitive probabilities?
5. How long could a set of dice reverse as the number of dice increase?

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About the Author



Nicholas Pasciuto is graduating in May 2016 with a major in Mathematics with a concentration in Pure Mathematics and a minor in Philosophy. His research project was completed in the summer of 2015 under the excellent mentorship of Dr. Ward Heilman (Mathematics). It was made possible with funding provided by an Adrian Tinsley Program summer research grant. Nicholas presented this paper at the 2015 Northeastern Section of the Mathematical Association of America and the 2016 National Conference on Undergraduate Research in Asheville, NC. He plans to pursue his Ph.D in Mathematics in the fall of 2016.